1. Topology of $\mathbb{R}^n$

Throughout, let $S \subseteq \mathbb{R}^n$. Let's recall some definitions from class.

Definition. *These definitions are related to subsets of* $\mathbb{R}^n$

(1) *S is called **open** if and only if $\forall\, x \in S$, there exists $\epsilon > 0$ such that $B(\epsilon, x) \subseteq S$*
(2) *S is called **closed** if and only if S^c is open*
(3) $\partial S = \{x \in \mathbb{R}^n \,|\, \forall\, \epsilon > 0\, B(\epsilon, x) \cap S \neq \emptyset \text{ and } B(\epsilon, x) \cap S^c \neq \emptyset\}$
(4) $\operatorname{int} S = \{x \in \mathbb{R}^n \,|\, \exists\, \epsilon > 0\, B(\epsilon, x) \subseteq S\}$
(5) *$\overline{S} = S \cup \partial S$ is called the **closure** of S. It is the smallest closed set containing S.*

Problem 1. *Show whether the following subsets of $\mathbb{R}^2$ are open, closed, neither, or both open and closed. Describe the interior, the closure, and the boundary.*

(1) $S = \emptyset$
(2) $S = \big\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\big\}$
(3) $S = \big\{(x, y) \in \mathbb{R}^2 \,|\, 0 < x^2 + y^2 \leq 1\big\}$

Part 1: $S = \emptyset$ is open. Given any $x \in S$, we have to produce an open ball around x completely contained in S. As there are no points to consider, the definition of open is vacuously true for the empty set. The empty set is also closed; $\emptyset^c = \mathbb{R}^2$ which is open. The interior, closure, and boundary of the empty set is still the empty set.

Part 2: The open unit ball is open. Pick any $(x, y) \in B(1, 0)$, then let d be the distance from (x, y) to the unit circle. $B((x, y), d/2) \subseteq B(1, 0)$ is a ball around (x, y) completely contained in $B(1, 0)$. As $B(1, 0)$ is open, we have $\operatorname{int} B(1, 0) = B(1, 0)$. The boundary of $B(1, 0)$ is the unit circle $\big\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\big\}$.

Part 3: This example is similar to the previous one, only now we have removed a point from the interior of the ball and added the boundary. The set is neither open nor closed. To see that it is not open, pick any point $p \in \big\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\big\}$, then fix any $\epsilon > 0$. The ball $B(\epsilon, p)$ necessarily contains points with $x^2 + y^2 > 1$, therefore no ball around p could be completely contained in S. To see that the set is not closed, notice that $S^c = \{(0, 0)\} \cup \big\{(x, y) \,|\, x^2 + y^2 > 1\big\}$. Any ball $B(\epsilon, (0, 0))$ around the origin must contain points in S; therefore, since S^c is not open, S is not closed. The interior is $\operatorname{int} S = \big\{(x, y) \,|\, 0 < x^2 + y^2 < 1\big\}$. The boundary of S is $\partial S = \{(0, 0)\} \cup \big\{(x, y) \,|\, x^2 + y^2 = 1\big\}$. The closure is $\overline{S} = S \cup \partial S = \big\{(x, y) \,|\, x^2 + y^2 \leq 1\big\}$.

Class Exercise: (Left side of the class) Repeat the previous problem for $S = \{(x, y) \,|\, x > 0 \text{ and } y = \sin(1/x)\}$. (Right side of the class) Find an example of a set S such that $\operatorname{int} S \neq \operatorname{int} \overline{S}$.

Solution: The set is neither closed nor open; to see that it is not closed, notice that any point in $\{(x, y) \,|\, x = 0 \text{ and } y \in [-1, 1]\}$ is in the boundary of S, and these points are not in S since $x > 0$ for all points in S. The interior of the set is empty.

One example of a set S such that $\operatorname{int} S \neq \operatorname{int} \overline{S}$ is $S = \mathbb{Q} \subseteq \mathbb{R}$. Here, $\operatorname{int} \mathbb{Q} = \emptyset$, but $\overline{\mathbb{Q}} = \mathbb{R}$ and the interior of $\mathbb{R}$ is $\mathbb{R}$.

Problem 2. *If S_1 and S_2 are open, so are $S_1 \cup S_2$ and $S_1 \cap S_2$.*

Remark: This problem commonly shows up on exams!

Let $S = S_1 \cup S_2$ and pick any $x \in S$, then $x \in S_1$ or $x \in S_2$. If $x \in S_1$, then since S_1 is open $\exists\, \epsilon > 0$ such that $B(\epsilon, x) \subseteq S_1 \subseteq S$. If $x \in S_2$ then an identical argument holds true; therefore, S is open.

If $S_1 \cap S_2$ is empty then it is open. Now suppose $S = S_1 \cap S_2$ is non empty and pick any $x \in S$, then $x \in S_1$ and $x \in S_2$. Since S_1 and S_2 are open, $\exists\, \epsilon_1, \epsilon_2 > 0$ such that $B(\epsilon_1, x) \subseteq S_1$ and $B(\epsilon_2, x) \subseteq S_2$. If we set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then $B(\epsilon, x) \subseteq S_1$ and $B(\epsilon, x) \subseteq S_2$, therefore $B(\epsilon, x) \subseteq S$. This shows that $S_1 \cap S_2$ is open.